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# On the Unit Group of a Semigroup Ring

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A submonoid  $S$  of a torsion-free abelian (additive) group is called a grading monoid (or a  $g$ -monoid). Throughout the paper we assume that  $S$  is non-zero.

We consider the semigroup ring  $R[X; S]$  of a  $g$ -monoid  $S$  over a commutative ring  $R$ .

We denote the unit group  $\{s \in S \mid s + t = 0 \text{ for some } t \in S\}$  of  $S$  by  $H = H(S)$ .

We denote the nilradical of  $R$ , that is, the set of nilpotents of  $R$ , by  $N = N(R)$ . If  $N = 0$ , then  $R$  is called reduced.

We denote the unit group of  $R$  by  $U = U(R)$ .

We denote the group of units  $f = \sum a_s X^s$  of  $R[X; S]$  with  $\sum a_s = 1$  by  $V(R[X; S])$ .  $H$  is canonically regarded as a subgroup of  $V(R[X; S])$ .

Let  $G$  be an abelian group. If  $G$  has only one elements, or if  $G$  has a free basis which is not necessarily of finite number, then  $G$  is called free. Any subgroup of a free group is free.

An element  $x$  of an abelian multiplicative group  $G$  is called torsion, if  $x^n = 1$  for some positive integer  $n$ . The set of torsion elements of  $G$  is a subgroup of  $G$ . If 1 is the only torsion elements of  $G$ , then  $G$  is called torsion-free.

The symbol  $\otimes$  denotes direct product of groups.

Karpilovsky posed 21 research problems in [K, Chapter 7]. The 9th problem is the following:

Let  $G$  be an abelian group. Find necessary and sufficient conditions for  $R[X; G]$  under which

- (1)  $G$  has a torsion-free complement in  $V(R[X; G])$ .
- (2)  $G$  has a free complement in  $V(R[X; G])$ .
- (3)  $U(R[X; G])$  is free modulo torsion.

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This is an abstract and the details will appear elsewhere.

In [M1] we posed its semigroup version as follows:

**Problem.** Let  $S$  be a g-monoid. Find necessary and sufficient conditions for  $R[X; S]$  under which

(1)  $H$  has a torsion-free complement in  $V(R[X; S])$ . That is, there exists a torsion-free subgroup  $W$  of  $V(R[X; S])$  such that  $V(R[X; S]) = H \otimes W$ .

(2)  $H$  has a free complement in  $V(R[X; S])$ . That is, there exists a free subgroup  $W$  of  $V(R[X; S])$  such that  $V(R[X; S]) = H \otimes W$ .

(3)  $U(R[X; S])$  is free modulo torsion. That is, the residue class group  $U(R[X; S])/\{f \in U(R[X; S]) \mid f \text{ is torsion}\}$  is free.

In §1, we review results in [M1, Section 1], [M2, Section 6] and [M3]. In §2, we give a preciser decomposition theorem for the unit group  $U(R[X; S])$  of  $R[X; S]$ . And, using the decomposition theorem, we give a reduction for Problem.

## §1. Review

Let  $E = \{e_\lambda \mid \lambda \in \Lambda\}$  be a set of non-zero idempotents of  $R$ . If, for each  $\lambda_1$  and  $\lambda_2$  of  $\Lambda$ , there exists  $\lambda_3 \in \Lambda$  such that  $e_{\lambda_3} \in Re_{\lambda_1} \cap Re_{\lambda_2}$ , then  $E$  is called an E-system of  $R$ . There exists a maximal (by inclusion) E-system of  $R$  by Zorn's Lemma.

Let  $E$  be a fixed maximal E-system of  $R$ . We set

$W(R[X; S]) = \{\sum a_s X^s \in V(R[X; S]) \mid Ra_0 \text{ contains an element of } E\}$ .

Then  $W(R[X; S])$  is a subgroup of  $V(R[X; S])$ .

**Proposition 1.** (1)  $V(R[X; S]) = H \otimes W(R[X; S])$ .

(2)  $U(R[X; S]) = U(R) \otimes H \otimes W(R[X; S])$ .

For the proof of Proposition 1(2), let  $f = \sum a_s X^s$  be a unit of  $R[X; S]$ . Then  $u = \sum a_s$  is a unit of  $R$ , and  $(1/u)f \in V(R[X; S])$ . It follows that  $U(R[X; S]) = U(R) \otimes V(R[X; S])$ . Then (1) completes the proof.

Proposition 1 (1) shows that  $W(R[X; S])$  does not depend on the maximal E-system of  $R$  up to isomorphisms.

We set  $M = M(R) = \{x \in N \mid nx = 0 \text{ for some positive integer } n\}$ .

**Theorem 1** (An answer to Problem (1)). The following conditions are equivalent:

- (1)  $H$  has a torsion-free complement in  $V(R[X; S])$ .
- (2)  $V(R[X; S])$  is torsion-free.
- (3)  $W(R[X; S])$  is torsion-free.
- (4)  $M = 0$ .

Set  $R' = R/M$ . For each maximal E-system  $E'$  of  $R'$ , we may define  $W(R'[X; S])$ .

**Theorem 2** (A reduction of Problem (3) to Problem (2)). The following conditions are equivalent:

- (1)  $U(R[X; S])$  is free modulo torsion.
- (2)  $U(R)$  is free modulo torsion, and  $V(R'[X; S])$  is free.
- (3)  $U(R)$  is free modulo torsion, and both  $H$  and  $W(R'[X; S])$  are free.
- (4)  $U(R)$  is free modulo torsion,  $H$  is free, and  $H$  has a free complement in  $V(R'[X; S])$ .

If  $R$  has only a finite number of idempotents, then  $R$  is called almost indecomposable. If 0 and 1 are only the idempotents of  $R$ , then  $R$  is called indecomposable. If  $R$  is not indecomposable, then  $R$  is called decomposable.

**Proposition 2** (A reduction of Problem for almost indecomposable rings to indecomposable rings). Let  $e_1, \dots, e_n$  be non-zero idempotents of  $R$  such that  $e_1 + \dots + e_n = 1$  and  $e_i e_j = 0$  for  $i \neq j$ , where  $n \geq 2$ .

Then,

- (1)  $H$  has a free complement in  $V(R[X; S])$ , if and only if  $H$  is free and, for each  $i$ ,  $H$  has a free complement in  $V(Re_i[X; S])$ .

(2)  $U(R[X; S])$  is free modulo torsion, if and only if, for each  $i$ ,  $U(Re_i[X; S])$  is free modulo torsion.

(3)  $U(R[X; S])$  is a finitely generated free abelian group modulo torsion, if and only if, for each  $i$ ,  $U(Re_i[X; S])$  is a finitely generated free abelian group modulo torsion.

**Theorem 3.** (1)  $H$  has a finitely generated free complement in  $V(R[X; S])$ , if and only if  $R$  is reduced and either  $R$  is indecomposable or  $H = 0$ .

(2)  $U(R[X; S])$  is a finitely generated free abelian group modulo torsion, if and only if  $U(R)$  is a finitely generated free abelian group modulo torsion,  $H$  is a finitely generated free abelian group,  $N = M$ , and either  $R$  is indecomposable or  $H = 0$ .

**Theorem 4** (An answer to Problem for reduced rings). Let  $R$  be reduced. Then,

(1)  $H$  has a torsion-free complement in  $V(R[X; S])$ .

(2)  $H$  has a free complement in  $V(R[X; S])$  if and only if  $H$  is free.

(3)  $U(R[X; S])$  is free modulo torsion, if and only if  $U(R)$  is free modulo torsion and  $H$  is free.

## §2. Results

Let  $E = \{e_\lambda \mid \lambda \in \Lambda\}$  be a fixed maximal E-system of  $R$ . Then  $E \ni 1$ . The characteristic of  $R$  is denoted by  $ch(R)$ .

**Proposition 3.** (1) Assume that  $N \neq 0$ , and assume that  $H$  has a free complement in  $V(R[X; S])$ . Then  $ch(R) = 0$ , and  $M = 0$ .

(2) Assume that  $N \neq 0$  and  $ch(R) > 0$ . Then  $U(R[X; S])$  is free modulo torsion, if and only if  $U(R)$  is free modulo torsion and  $H$  is free.

(1) follows from Theorem 1, and the necessity of (2) follows from Theorem 2.

For the sufficiency of (2), let  $R' = R/M$ . Then  $R'$  is reduced. By Theorem 4,  $H$  has a free complement in  $V(R'[X; S])$ . By Theorem 2,

$U(R[X; S])$  is free modulo torsion.

**Proposition 4.** (1) Assume that  $N \neq 0$  and  $R \supset \mathbf{Q}$ , where  $\mathbf{Q}$  is the field of rational numbers. Then  $H$  does not have a free complement in  $V(R[X; S])$ .

(2) Assume that  $N \neq 0$  and  $R \supset \mathbf{Q}$ . Then  $U(R[X; S])$  is not free modulo torsion.

For the proof of (1), take a non-zero element  $x_0 \in R$  such that  $x_0^2 = 0$ , and take a non-zero element  $s_0 \in S$ . Set  $W_1 = \{1 + \alpha x_0 - \alpha x_0 X^{s_0} \mid \alpha \in \mathbf{Q}\}$ . Then  $W_1$  is a subgroup of  $W(R[X; S])$ , and is isomorphic onto the additive group  $\mathbf{Q}$ . If  $W(R[X; S])$  is free, and hence  $W_1$  is free, then  $\mathbf{Q}$  is free; a contradiction.

(2) Then  $M = 0$ , and hence  $R' = R$ . By (1),  $H$  does not have a free complement in  $V(R[X; S])$ . By Theorem 2,  $U(R[X; S])$  is not free modulo torsion.

**Proposition 5.**  $U(R[X; S])$  is free modulo torsion if and only if  $U(R'[X; S])$  is free modulo torsion.

For the proof, let  $U$  be the unit group of  $R$ , and let  $T$  be the set of torsion elements of  $U$ . Let  $U'$  be the unit group of  $R' = R/M$ , and let  $T'$  be the set of torsion elements of  $U'$ . We can show that  $U' = \{\bar{u} \mid u \in U\}$ , and  $T' = \{\bar{t} \mid t \in T\}$ . Moreover, we can show that  $U/T \cong U'/T'$ . Then the proof follows from Theorem 2.

Proposition 5 reduces Problem (3) to the case where  $M = 0$ .

**Lemma 1.** Let  $E = \{e_\lambda \mid \lambda \in \Lambda\}$  be a fixed maximal E-system of  $R$ . Let  $W_\Lambda = W_\Lambda(R[X; S])$  be the subgroup of  $W(R[X; S])$  generated by its subset  $\{e_\lambda + e'_\lambda X^\alpha \mid \lambda \in \Lambda, \alpha \in H\}$ , where  $e'_\lambda = 1 - e_\lambda$ . Let  $W_N = W_N(R[X; S]) = \{\sum a_s X^s \in W(R[X; S]) \mid Ra_0 \text{ contains } 1\}$ . Then  $W(R[X; S]) = W_\Lambda(R[X; S]) \otimes W_N(R[X; S])$ .

To show that  $W_\Lambda \cap W_N = 1$ , assume that  $W_\Lambda \cap W_N \ni f = \prod_1^n (e_{i1} + e_{i2}X^{\alpha_i}) = 1 + x_0 + x_1X^{s_1} + \dots + x_mX^{s_m}$ , where  $e_{i1} \in E, e_{i2} = 1 - e_{i1}, \alpha_i \in H$ , and  $0 \neq x_i \in N$  for  $i \geq 1$ . If  $m = 0$ , then  $f = 1$ . If  $m \geq 1$ , and if  $(i_1, \dots, i_n) \neq (j_1, \dots, j_n)$ , then  $e_{1i_1}e_{2i_2} \dots e_{ni_n}$  and  $e_{1j_1}e_{2j_2} \dots e_{nj_n}$  are orthogonal. It follows that  $x_1$  is an idempotent; a contradiction.

Let  $f = \epsilon_0 + \epsilon_1X^{\alpha_1} + \dots + \epsilon_nX^{\alpha_n} + x_1X^{s_1} + \dots + x_mX^{s_m}$  be an element of  $W(R[X; S])$ , where  $\epsilon_i$  is non-zero idempotent,  $\alpha_i \in H, x_i \in N$ . We can show that  $f \in W_\Lambda \otimes W_N$  by induction on  $n$ .

**Lemma 2.** Let  $W_H = W_H(R[X; S]) = \{\sum a_\alpha X^\alpha \in W_N(R[X; S]) \mid \alpha \in H\}$ , and  $W_m = W_m(R[X; S]) = \{\sum a_s X^s \in W_N \mid s \text{ is } 0 \text{ or a non-unit of } S\}$ . Then  $W_N = W_H \otimes W_m$ .

For the proof, let  $f \in W_N$ .  $f$  may be written as  $f = \sum a_i X^{\alpha_i} + \sum b_j X^{s_j}$ , where  $\alpha_i \in H$  and  $s_j$  is a non-unit of  $S$ . Set  $f_1 = \sum a_i X^{\alpha_i}$  and  $f_2 = \sum b_j X^{s_j}$ . Then  $f_1$  is a unit of  $R[X; S]$ . Set  $g_1 = \sum c_k X^{t_k} = 1 + f_1^{-1}f_2, u_1 = \sum a_i, v_1 = \sum c_k$ . Then  $u_1v_1 = 1, f = (v_1f_1)(u_1g_1), v_1f_1 \in W_H$  and  $u_1g_1 \in W_m$ .

**Theorem 5.** (1)  $W(R[X; S]) = W_\Lambda(R[X; S]) \otimes W_H(R[X; S]) \otimes W_m(R[X; S])$ .

(2)  $W_\Lambda(R[X; S]) = 1$ , if and only if either  $R$  is indecomposable or  $H = 0$ .

(3)  $W_H = W_m = 1$ , if and only if  $W_N = 1$ , if and only if  $R$  is reduced.

(4) Both  $W_H = 1$  and  $W_m \neq 1$ , if and only if both  $N \neq 0$  and  $H = 0$ .

(5) Both  $W_H \neq 1$  and  $W_m = 1$ , if and only if both  $N \neq 0$  and  $H = S$ .

**Lemma 3.** Assume that  $R$  is decomposable. Then  $W_\Lambda(R[X; S])$  is free if and only if  $H$  is free.

For the proof, let  $E = \{e_\lambda \mid \lambda \in \Lambda\}$  be a maximal E-system of  $R$  with  $e_0 = 1$ . Let  $e = e_\lambda$  with  $\lambda \neq 0$ . Then  $W_1 = \{e + e'X^\alpha \mid \alpha \in H\}$  is a subgroup of  $W_\Lambda$ , and  $W_1 \cong H$ . If  $W_\Lambda$  is free, then  $W_1$  is free, and hence  $H$  is free.

The converse follows from [M3, §2, Proposition 1].

**Proposition 6.** Let  $S_1 = (S - H) \cup \{0\}$ . Then  $W(R[X; S])$  is free, if and only if both  $W(R[X; H])$  and  $W(R[X; S_1])$  are free.

For the proof, denote  $V(S) = V(R[X; S])$ ,  $W_\Lambda(S) = W_\Lambda(R[X; S])$ , and etc. We have;

$$V(S) = H \otimes W_\Lambda(S) \otimes W_H(S) \otimes W_m(S).$$

$$V(H) = H \otimes W_\Lambda(H) \otimes W_H(H).$$

$$V(S_1) = W_\Lambda(S_1) \otimes W_m(S_1).$$

$$W_H(S) \cong W_H(H).$$

$$W_m(S) \cong W_m(S_1).$$

The proof follows from the above formulas.

Proposition 6 shows that Problem (2) reduces to the case where either every elements of  $S$  is unit, or every element of  $S$  is non-unit except zero.

Proposition 6 implies the following,

**Proposition 7.** Let  $S_1 = (S - H) \cup \{0\}$ . Then  $U(R[X; S])$  is free modulo torsion, if and only if both  $U(R[X; H])$  and  $U(R[X; S_1])$  are free modulo torsion.

Proposition 7 shows that Problem (3) reduces to the case where either every elements of  $S$  is unit, or every element of  $S$  is non-unit except zero.

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